Mapping and Embedding of Two Metrics Associated with Dark Matter, Dark Energy, and Ordinary Matter

A. C. V. V. de Siqueira *
Departamento de Educação
Universidade Federal Rural de Pernambuco
52.171-900, Recife, PE, Brazil.

Abstract

In this paper we build a mapping between two different metrics and embed them in a flat manifold. One of the metrics represents the ordinary matter, and the other describes the dark matter, the dark energy, and the particle-antiparticle asymmetry.The latter was obtained in a recent paper. For the mapping and embedding we use two new formalisms developed and presented in two previous papers, **Mapping Among Manifolds** and, **Conformal Form of Pseudo-Riemannian Metrics by Normal Coordinate Transformations**, which was a generalization of the Cartan's approach of Riemannian normal coordinates.

* E-mail: acvvs@ded.ufrpe.br

1 Introduction

The Einstein's theory of general relativity is still the best theory to describe problems in astrophysics and cosmology. However, more recent observations in these two areas are apparently difficult to be explained by general relativity. It raises the possibility to consider models involving membranes and parallel universes, dark matter, dark energy, and the cosmological constant to explain the behavior of large scale structures like galaxies, clusters of galaxies, and the universe. In a previous paper we obtained a spatially flat solution of the Einstein's equation with a Klein-Gordon field and the cosmological constant [1]. We have shown in a recent paper [2], that this scalar particle solution is a candidate to explain the possible origin of dark matter, dark energy, and particle-antiparticle asymmetry. The introduction of a new field in the Einstein's equation could destroy our dark matter and dark energy solution. If it is a good candidate to describe dark matter, dark energy, and particle-antiparticle asymmetry, then the Einstein's equation needs to be preserved as in [2]. The embedding of one metric only in an n-dimensional flat space is well known. The embedding of a classical metric and the dark matter and dark energy metric in an n-dimensional flat space is a possible strategy in order to consider the gravitational interaction between the scalar particle and the ordinary matter.
This paper is organized as follows. In Sec. 2, we present a primordial and spatially flat solution of the Einstein's equation with a massive scalar Klein-Gordon field and the cosmological constant. The Jacobi equation is presented for this solution and two primordial forces are identified as dark matter and dark energy, respectively. In Sec. 3, we present some of our results in **Conformal Form of Pseudo-Riemannian Metrics by Normal Coordinate Transformations** as a generalization of the Cartan's approach of Riemannian normal coordinates [3]. In Sec. 4, we present some of our results in **Mapping Among Manifolds** [4]. As we will see, these are important for the embedding of two different manifolds in an n-dimensional flat manifold. In Sec. 5, we build a mapping between dark matter, dark energy and ordinary matter. In Sec. 6, we build an embedding of two metrics, one associated with ordinary matter and the other with dark matter and dark energy. In Sec. 7, we present the two 6-dimensional hyper-vectors, normal to the dark matter-dark energy manifold, and show how to build two 6-dimensional

hyper-vectors, normal to the ordinary matter manifold, and how to make the mapping among 6-dimensional hyper-vectors. In Sec.8, we summarize and conclude the results of this paper.

2 An Exact Solution of the Einstein's Equation

In a previous paper we obtained three solutions of the Einstein's equation with a Klein-Gordon field and the cosmological constant [1]. In a recent paper we presented in details some aspects of the spatially flat solution relevant to dark matter, dark energy, and particle-antiparticle asymmetry [2]. In this paper it will be necessary to introduce some of those aspects as a short review. The convention used in a local basis was [1], [2],

$$R^{\alpha}{}_{\mu\sigma\nu} = \partial_\nu \Gamma^{\alpha}_{\mu\sigma} - \partial_\sigma \Gamma^{\alpha}_{\mu\nu} + \Gamma^{\eta}_{\mu\sigma}\Gamma^{\alpha}_{n\nu} - \Gamma^{\eta}_{\mu\nu}\Gamma^{\alpha}_{\sigma\eta} \tag{2.1}$$

with Ricci tensor

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}. \tag{2.2}$$

For this convention we have the following Einstein's equation, with the cosmological constant Λ,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^2} T_{\mu\nu} \tag{2.3}$$

where $T_{\mu\nu}$ is the momentum-energy tensor of a massive scalar field,

$$T_{\mu\nu} = 2\nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi + m^2 g_{\mu\nu} \phi^2 \tag{2.4}$$

We have used (+, -, -, -) signature convention and a Friedmann-Robertson-Walker line element given by

$$ds^2 = dt^2 - \frac{d\sigma^2 e^g}{(1 + Br^2)^2} \tag{2.5}$$

where $d\sigma^2$ is the three-dimensional Euclidian line element and $A = 8\pi G/c^2$, $B = k/4a'^2$ and with $k = 0$, $k = 1$ and $k = -1$. We also have a'^2 as a constant.

We pay attention to our spatial flat solution, $B = 0$. In this case the field is given by

$$\phi = \frac{\in mt}{\sqrt{3A}} + b \tag{2.6}$$

with $\in = \pm 1$ and b as an arbitrary constant. The cosmological constant obeys the condition

$$\Lambda = -\frac{m^2}{3} = -\frac{1}{3}(\frac{cM}{\hbar})^2, \tag{2.7}$$

a negative value, associated with the Planck's constant, the speed of light and a scalar particle of mass M.
The corresponding line element is

$$ds^2 = dt^2 - d\sigma^2 e^{[-2\in mb(\sqrt{\frac{A}{3}})t - \frac{m^2}{3}t^2]}. \tag{2.8}$$

In this paper we use another convention to the Riemann tensor, as follows,

$$R^{\alpha}{}_{\mu\sigma\nu} = -\partial_\nu \Gamma^{\alpha}_{\mu\sigma} + \partial_\sigma \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\eta}_{\mu\sigma}\Gamma^{\alpha}_{n\nu} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\alpha}_{\sigma\eta} \tag{2.9}$$

which implies

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^2}T_{\mu\nu}. \tag{2.10}$$

The motion will be simpler in a Fermi-Walker transported tetrad basis.
Let us consider the connection between the tetrad and the local metric tensor

$$g_{\lambda\pi} = E^{(\mathbf{A})}_{\lambda} E^{(\mathbf{B})}_{\pi} \eta_{(\mathbf{A})(\mathbf{B})}, \tag{2.11}$$

where $\eta_{(\mathbf{A})(\mathbf{B})}$ and $E^{(\mathbf{A})}_{\lambda}$ are the Lorentzian metric and tetrad components, respectively.

From (2.8) we have

$$E_0^{(\mathbf{0})} = 1, \tag{2.12}$$

$$E_1^{(\mathbf{1})} = E_2^{(\mathbf{2})} = E_3^{(\mathbf{3})} = e^{[-\in mb(\sqrt{\frac{A}{3}})t - \frac{m^2}{6}t^2]}. \tag{2.13}$$

We now write the 1-form

$$\theta^{(\mathbf{A})} = dx^\lambda E_\lambda^{(\mathbf{A})}. \tag{2.14}$$

By exterior derivatives of (2.14) and using the Cartan's second structure equation, we obtain

$$R^{(\mathbf{1})}{}_{(\mathbf{0})(\mathbf{0})(\mathbf{1})} = R^{(\mathbf{2})}{}_{(\mathbf{0})(\mathbf{0})(\mathbf{2})} =$$

$$= R^{(\mathbf{3})}{}_{(\mathbf{0})(\mathbf{0})(\mathbf{3})} = -\frac{m^2}{3} + \frac{1}{2}[-2 \in mb(\sqrt{\frac{A}{3}}) - \frac{m^2}{3}t]^2. \tag{2.15}$$

Let us present the Jacobi equation in a Fermi-Walker transported tetrad basis [6],

$$\frac{d^2 Z^{(\mathbf{A})}}{dt^2} + R^{(\mathbf{A})}{}_{(\mathbf{0})(\mathbf{C})(\mathbf{0})} Z^{(\mathbf{C})} = 0. \tag{2.16}$$

Substituting (2.15) in (2.16) we obtain

$$\frac{d^2 Z^{(\mathbf{A})}}{dt^2} = \{-\frac{m^2}{3} + \frac{1}{2}[-2 \in mb(\sqrt{\frac{A}{3}}) - \frac{m^2}{3}t]^2\} Z^{(\mathbf{A})}, \tag{2.17}$$

with $A = (1, 2, 3)$.
We can rewrite (2.17) as follows

$$\frac{d^2 Z^{(\mathbf{A})}}{dt^2} = \{-\frac{m^2}{3} - 2bm^3\frac{1}{3}(\sqrt{\frac{A}{3}})t + \frac{m^2}{3}b^2 A + \frac{m^4}{18}t^2\} Z^{(\mathbf{A})}. \tag{2.18}$$

The Jacobi equation will be appropriate to show the relative acceleration between two particles if we do not have to consider the metric deformation by particles. For the primordial universe (2.8), we have from (2.17) or (2.18) that two massive scalar particles in two geodesics close to each other feel two primordial forces, being one attractive (dark matter) and the other repulsive (dark energy), both increasing with distance. The same scalar particle will be responsible for the two metric forces which, conveniently, we have identified as dark matter and dark energy. From the gravitational point of view, the

creation of other types of matter by the universe generates three competitive forces, the two primordial forces above presented, and another which, for galaxies, can be expressed by the Newtonian gravity. Inside and outside the galaxies, the resulting force is the sum of these three forces. A correct dynamic description of one or more stars in a galaxy, depends on a set of information about galaxy evolution. Elliptical and spiral galaxies, as well as clusters of galaxies, have different dynamics and different evolution processes. The Newtonian gravity is very important to describe the galaxies dynamics but it is not enough. Physicists and astronomers have concluded that the Newtonian gravity only is not sufficient to describe the galaxies dynamics. They believe in the existence of a second attractive force (dark matter) which, in association with the Newtonian gravity, governs the star dynamics. They also believe in the existence of a repulsive force (dark energy) responsible for the expansion on large scale. We believe that the presence of the two primordial forces together with the Newtonian force can describe the galaxies behavior. Inside and outside the galaxies the resulting force is the sum of the three forces. The value of the constant b in (2.6) and (2.8) could be fixed by experimental records of galaxies (dark matter) or cosmological expansion (dark energy) or both. Modifications in the stars motion in galaxies can be done by appropriate adjustments in the constant b. It is possible that b is a new constant of nature, as well as the mass M of the scalar particle.

The mass of the scalar particle can be estimated by astronomic measurements of the cosmological constant. From (2.7) and (2.17) we conclude that Λ is associated with an attractive force, for us conveniently identified as dark matter, and a repulsive force, identified as dark energy. The second term in the second member of (2.17) is positive and, therefore, identified as dark energy. Notice that the dark energy term is a function of time, of the constant b, and also of the term identified as dark matter. We noticed in (2.18) that the constants m and b are present in the terms associated with the dark matter (attractive force) and to the terms associated with the dark energy (repulsive force). Then, Λ is present in the dark matter and in the dark energy. In other words, it is not possible to separate them, because dark matter and dark energy are scalar particle effects. Originally, the cosmological constant was associated with a repulsive force so that the estimate given in the following is associated with a large scale expansion. It can be slightly different from a realistic and definitive value, but our objective is to point that we cannot detect the scalar particle, or, at best, we have a very low probability of doing

so.
Using the experimental limit for the constant Λ in (2.7) [6], it is possible to obtain a superior limit for the mass of the scalar particle. The cosmological constant was estimated as

$$\Lambda < 10^{-54} cm^{-2}. \tag{2.19}$$

Using it and (2.7) we obtain

$$M < (6).10^{-65} g. \tag{2.20}$$

There is another limit for the cosmological constant [7] given by

$$\Lambda Lp^2 < 10^{-123}, \tag{2.21}$$

or

$$\Lambda < 10^{-57} cm^{-2}. \tag{2.22}$$

Using it and (2.7) we obtain

$$M < (1.9).10^{-66} g. \tag{2.23}$$

The relationship between the electron rest mass and the mass of the scalar particle is approximately given by

$$m_e \sim (4.79).10^{38} M. \tag{2.24}$$

The universe expansion can be calculated. In other words, it is possible to calculate the starting point of the universe contraction. Using (2.31) and (2.8) we obtain

$$t \sim \frac{1}{\sqrt{-\Lambda}}. \tag{2.25}$$

Notice that we have assumed $c = 1$ in (2.8). Therefore, for numeric results, involving time, we regain ct, so that

$$t \sim 10^{27} \frac{cm}{c} \sim (3.33) 10^9 years, \tag{2.26}$$

where (2.19) and (2.25) were used.
Using (2.22) in (2.25) we have

$$t \sim (3.162) 10^{28} \frac{cm}{c} \sim (33.4) 10^9 years. \tag{2.27}$$

Note that (2.26) is incompatible with the geological data of the Earth. If (2.27) is a good estimate, it will be almost impossible to detect the scalar particle. Its influence will be predominantly gravitational and it is given by (2.8). Consequently, for many classical situations as, for instance, the solar system dynamics, the effect on the ordinary matter would be insignificant. For this condition we consider only the ordinary matter in the Einstein's equation. The scalar particle can be very important for galaxies, clusters of galaxies, and large structures. It is necessary a investigation to evaluate the influence of an intense gravitational field generated by a classical black hole geometry on the primordial scalar particle.

The primordial universe (2.8) starts with scalar particles and is non-singular at $t = 0$. It is an expansible universe if the curvature $R^{(A)}{}_{(A)}$ obeys a simple inequality. With the time evolution, other types of matter were created and complex interactions among particles are checked every day. Analytical solutions of (2.10) with the inclusion of other fields are very difficult. However, as the influence of the primordial universe (2.8) could have been very important in the past and can be very important in the present, it is reasonable to suppose that other primordial particles are ghosts, so that (2.8) is a consequence of the scalar particles only. In other words, the momentum-energy tensors of other primordial fields have not contributed to the curvature of the primordial universe in the past nor in the present, although such particles interact with all that, playing an important part in the evolution of the universe, as well as in the creation of the ordinary matter. We recall that in the cosmological models, metrics as the Friedmann-Robertson-Walker are important to the initial large structure formation, as well as to the universe evolution. But, gradually each local distribution of matter will be more and more important and the effect of all distributions of matter in the universe is represented by a momentum-energy tensor of a fluid in the Einstein's equation for a Friedmann-Robertson-Walker metric. However, if (2.8) is responsible for the dark matter and the dark energy, we will have a different situation. In this case (2.8) would determine the evolution of the universe in the past and in the present and, due to the mass estimate of the scalar particle, its interaction with other particles would be predominantly gravitational.

It is important to notice the presence of two different time scales, one associated with a local distribution, as well as with a large scale structure of ordinary matter, and the other associated with the cosmological time of (2.8). The embedding of (2.8) and of a classical metric in an n dimensional

flat space is a possible strategy to consider the gravitational interaction between the scalar particles and the ordinary matter.
The primordial universe (2.8) is non-singular at $t = 0$. It is cyclical and eternal, and could have different cycles. Although it is not the only possibility, a negative curvature is the simplest mechanism for an expansive universe and it will be considered.
For (2.8) the curvature is given by

$$R^{(A)}{}_{(A)} =$$
$$= -2(\frac{cM}{\hbar})^2 + 6[-2 \in \frac{cM}{\hbar} b(\sqrt{\frac{A}{3}}) - \frac{(\frac{cM}{\hbar})^2}{3} t]^2. \tag{2.28}$$

We have at $t = 0$

$$R^{(A)}{}_{(A)}(t = 0) =$$
$$= 2(\frac{cM}{\hbar})^2[-1 + 4b^2.A], \tag{2.29}$$

which is a finite curvature. We consider an initial negative curvature $R^{(A)}{}_{(A)}$ as the simplest condition for the primordial expansive universe

$$R^{(A)}{}_{(A)}(t = 0) < 0, \tag{2.30}$$

so that

$$\|b\| < \frac{1}{2\sqrt{A}}, \tag{2.31}$$

or

$$\|b\| < \frac{c}{2\sqrt{8\pi G}}, \tag{2.32}$$

or

$$\|b\| < 10^{15}\sqrt{\frac{g}{cm}}, \tag{2.33}$$

where (2.33) is a superior limit for b. We have obtained two superior limits for the mass of the scalar particle and for the constant b given by (2.20) and (2.33), respectively. Note that b and M can be two new constants of nature. For (2.22), M will be given by (2.23), smaller than (2.20), reinforcing the previous conclusion that it is very difficult to detect this scalar particle.
Note that our choice of an initial negative curvature, as the expansion mechanism, imposed a superior limit for the constant b. However, other mechanisms are possible, so that the constant b can assume another limit, compatible with experimental results.

3 Conformal Form of a Pseudo-Riemannian Metric by Normal Coordinate Transformations

In a previous [3] paper we extended the Cartan's approach of Riemannian normal coordinates and showed that all n-dimensional pseudo-Riemannian metrics are conformal to an n-dimensional flat manifold, as well as to an n-dimensional manifold of constant curvature, when, in normal coordinates, they are well-behaved in the origin and in its neighborhood. As a consequence of geometry, without postulates, we obtained the classical and the quantum angular momenta of a particle . In this Section a short review of this approach will be presented.
Let us consider the line element

$$ds^2 = G_{\Lambda\Pi} du^\Lambda du^\Pi, \tag{3.1}$$

with

$$G_{\Lambda\Pi} = E^{(\mathbf{A})}_\Lambda E^{(\mathbf{B})}_\Pi \eta_{(\mathbf{A})(\mathbf{B})}, \tag{3.2}$$

where $\eta_{(\mathbf{A})(\mathbf{B})}$ and $E^{(\mathbf{A})}_\Lambda$ are flat metric and vielbein components, respectively. We choose each $\eta_{(\mathbf{A})(\mathbf{B})}$ as plus or minus Kronecker's delta function, where a Lorentzian metric signature will be a particular case.
Let us give the 1-form $\omega^{(\mathbf{A})}$ by

$$\omega^{(\mathbf{A})} = du^\Lambda E^{(\mathbf{A})}_\Lambda. \tag{3.3}$$

We now define Riemannian normal coordinates by

$$u^{\Lambda} = v^{\Lambda} t. \tag{3.4}$$

Substituting in (3.3)

$$\omega^{(\mathbf{A})} = t dv^{\Lambda} E_{\Lambda}^{(\mathbf{A})} + dt v^{\Lambda} E_{\Lambda}^{(\mathbf{A})}. \tag{3.5}$$

Let us define

$$z^{(\mathbf{A})} = v^{\Lambda} E_{\Lambda}^{(\mathbf{A})}, \tag{3.6}$$

so that

$$\omega^{(\mathbf{A})} = dt z^{(\mathbf{A})} + t dz^{(\mathbf{A})} + t E^{\Pi(\mathbf{A})} \frac{\partial E_{\Pi(\mathbf{B})}}{\partial z^{(\mathbf{C})}} z^{(\mathbf{B})} dz^{(\mathbf{C})}. \tag{3.7}$$

We now make

$$A^{(\mathbf{A})_{(\mathbf{B})(\mathbf{C})}} = t E^{\Pi(\mathbf{A})} \frac{\partial E_{\Pi(\mathbf{B})}}{\partial z^{(\mathbf{C})}}, \tag{3.8}$$

then

$$\varpi^{(\mathbf{A})} = t dz^{(\mathbf{A})} + A^{(A)_{(\mathbf{B})(\mathbf{C})}} z^{(\mathbf{B})} dz^{(\mathbf{C})}, \tag{3.9}$$

with

$$\omega^{(\mathbf{A})} = dt z^{(\mathbf{A})} + \varpi^{(\mathbf{A})}. \tag{3.10}$$

We have at $t = 0$

$$A^{(A)_{(\mathbf{B})(\mathbf{C})}}(t = 0, z^{(\mathbf{D})}) = 0, \tag{3.11}$$

$$\varpi^{(\mathbf{A})}(t = 0, z^{(\mathbf{D})}) = 0, \tag{3.12}$$

and

$$\omega^{(\mathbf{A})}(t = 0, z^{(\mathbf{D})}) = dt z^{(\mathbf{A})}. \tag{3.13}$$

Consider, at an n+1-dimensional manifold, a coordinate system given by $(t, z^{(\mathbf{A})})$. For each value of t we have a hyper-surface, where $dt = 0$ on each of them. We are interested in the hyper-surface with $t = 1$, where we verify the following equality

$$\omega^{(\mathbf{A})}(t = 1, z) = \varpi^{(\mathbf{A})}(t = 1, z). \tag{3.14}$$

From the above results [3],

$$\frac{\partial^2 (A_{(\mathbf{A})(\mathbf{C})(\mathbf{D})})}{\partial (t^2)} = t z^{(\mathbf{B})} R_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} + z^{(\mathbf{L})} z^{(\mathbf{M})} R_{(\mathbf{A})(\mathbf{L})(\mathbf{M})(\mathbf{N})} A_{(\mathbf{P})(\mathbf{C})(\mathbf{D})} \eta^{(\mathbf{N})(\mathbf{P})}. \tag{3.15}$$

Using the curvature symmetries we have the following solution

$$A_{(\mathbf{A})(\mathbf{C})(\mathbf{D})} + A_{(\mathbf{A})(\mathbf{D})(\mathbf{C})} = 0, \tag{3.16}$$

that is true for all t.
Then,

$$A_{(\mathbf{A})(\mathbf{C})(\mathbf{D})} = -A_{(\mathbf{A})(\mathbf{D})(\mathbf{C})}, \tag{3.17}$$

so that, we can rewrite (3.9) as

$$\varpi^{(\mathbf{A})} = tdz^{(\mathbf{A})} + \frac{1}{2}A^{(A)}{}_{(\mathbf{B})(\mathbf{C})}(z^{(\mathbf{B})}dz^{(\mathbf{C})} - z^{(\mathbf{C})}dz^{(\mathbf{B})}). \tag{3.18}$$

Let us define

$$A_{(\mathbf{A})(\mathbf{C})(\mathbf{D})} = z^{(\mathbf{B})}B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})}. \tag{3.19}$$

The following result is obtained by substituting (3.19) in (3.15),

$$\frac{\partial^2(B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})})}{\partial(t^2)} = tR_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} + z^{(\mathbf{L})}z^{(\mathbf{M})}R_{(\mathbf{A})(\mathbf{B})(\mathbf{L})(\mathbf{N})}B_{(\mathbf{P})(\mathbf{M})(\mathbf{C})(\mathbf{D})}\eta^{(\mathbf{N})(\mathbf{P})}. \tag{3.20}$$

Using the curvature symmetries we obtain the solution

$$B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} + B_{(\mathbf{B})(\mathbf{A})(\mathbf{C})(\mathbf{D})} = const., \tag{3.21}$$

for all t.
We can obtain

$$B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} + B_{(\mathbf{B})(\mathbf{A})(\mathbf{C})(\mathbf{D})} = 0. \tag{3.22}$$

In the following, for future use, we present the line element on the hypersurface

$$ds'^2 = \eta_{(\mathbf{A})(\mathbf{B})}\varpi^{(\mathbf{A})}\varpi^{(\mathbf{B})}. \tag{3.23}$$

We conclude that $B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})}$ has the same symmetries of the Riemann curvature tensor

$$B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} = -B_{(\mathbf{B})(\mathbf{A})(\mathbf{C})(\mathbf{D})} = -B_{(\mathbf{A})(\mathbf{B})(\mathbf{D})(\mathbf{C})}. \tag{3.24}$$

Using (3.19) and (3.24) we have

$$\begin{aligned} A_{(\mathbf{A})(\mathbf{C})(\mathbf{D})}dz^{(\mathbf{A})}z^{(\mathbf{C})}dz^{(\mathbf{D})} = \\ +\frac{1}{4}B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})}. \\ .(z^{(\mathbf{B})}dz^{(\mathbf{A})} - z^{(\mathbf{A})}dz^{(\mathbf{B})}). \\ .(z^{(\mathbf{C})}dz^{(\mathbf{D})} - z^{(\mathbf{D})}dz^{(\mathbf{C})}). \end{aligned} \tag{3.25}$$

By direct use of (3.23), (3.25), and (3.18) we have

$$\begin{aligned} ds'^2 = t^2 \eta_{(\mathbf{A})(\mathbf{B})} dz^{(\mathbf{A})} dz^{(\mathbf{B})} + \\ + \frac{1}{2}\{ \frac{1}{2} t \epsilon_{(\mathbf{B})} B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} + \\ + \eta^{(\mathbf{M})(\mathbf{N})} A_{(\mathbf{M})(\mathbf{B})(\mathbf{A})} A_{(\mathbf{N})(\mathbf{C})(\mathbf{D})} \}. \\ .(z^{(\mathbf{B})} dz^{(\mathbf{A})} - z^{(\mathbf{A})} dz^{(\mathbf{B})})(z^{(\mathbf{C})} dz^{(\mathbf{D})} - z^{(\mathbf{D})} dz^{(\mathbf{C})}). \end{aligned} \tag{3.26}$$

The line elements of the manifold and the hyper-surface are equal at $t = 1$, where $u^\Lambda = v^\Lambda$,

$$ds^2 = ds'^2, \tag{3.27}$$

and

$$\begin{aligned} ds^2 = \eta_{(\mathbf{A})(\mathbf{B})} dz^{(\mathbf{A})} dz^{(\mathbf{B})} + \\ + \frac{1}{2}\{ \frac{1}{2} \epsilon_{(\mathbf{B})} B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} + \\ + \eta^{(\mathbf{M})(\mathbf{N})} A_{(\mathbf{M})(\mathbf{B})(\mathbf{A})} A_{(\mathbf{N})(\mathbf{C})(\mathbf{D})} \}. \\ .(z^{(\mathbf{B})} dz^{(\mathbf{A})} - z^{(\mathbf{A})} dz^{(\mathbf{B})})(z^{(\mathbf{C})} dz^{(\mathbf{D})} - z^{(\mathbf{D})} dz^{(\mathbf{C})}). \end{aligned} \tag{3.28}$$

It can also be written in the form

$$
\begin{aligned}
&[1-\frac{1}{2}[\frac{1}{2}\epsilon_{(\mathbf{B})}B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} + \\
&+\eta^{(\mathbf{M})(\mathbf{N})}A_{(\mathbf{M})(\mathbf{B})(\mathbf{A})}A_{(\mathbf{N})(\mathbf{C})(\mathbf{D})}]. \\
&.(z^{(\mathbf{B})}\frac{dz^{(\mathbf{A})}}{ds}-z^{(\mathbf{A})}\frac{dz^{(\mathbf{B})}}{ds})(z^{(\mathbf{C})}\frac{dz^{(\mathbf{D})}}{ds}-z^{(\mathbf{D})}\frac{dz^{(\mathbf{C})}}{ds})]ds^2 \\
&=\eta_{(\mathbf{A})(\mathbf{B})}dz^{(\mathbf{A})}dz^{(\mathbf{B})}.
\end{aligned}
\tag{3.29}
$$

We now define the function

$$
L^{(\mathbf{A})(\mathbf{B})}=(z^{(\mathbf{A})}\frac{dz^{(\mathbf{B})}}{ds}-z^{(\mathbf{B})}\frac{dz^{(\mathbf{A})}}{ds}), \tag{3.30}
$$

which is the classical angular momentum of a free particle.
The line element (3.29) can assume the following form

$$
\begin{aligned}
&\{1+\frac{1}{2}[\frac{1}{2}(\epsilon_{(\mathbf{B})}B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} + \\
&+\eta^{(\mathbf{M})(\mathbf{N})}A_{(\mathbf{M})(\mathbf{B})(\mathbf{A})}A_{(\mathbf{N})(\mathbf{C})(\mathbf{D})}]. \\
&.(L^{\mathbf{A})(\mathbf{B})}L^{\mathbf{C})(\mathbf{D})})\}ds^2 \\
&=(\eta_{(\mathbf{A})(\mathbf{B})}dz^{(\mathbf{A})}dz^{(\mathbf{B})}.
\end{aligned}
\tag{3.31}
$$

We now define the function

$$
\begin{aligned}
\exp(-2\sigma)=&\{1+\frac{1}{2}[\frac{1}{2}(\epsilon_{(\mathbf{B})}B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} \\
&+\eta^{(\mathbf{M})(\mathbf{N})}A_{(\mathbf{M})(\mathbf{B})(\mathbf{A})}A_{(\mathbf{N}))(\mathbf{C})(\mathbf{D})})]. \\
&.L^{(\mathbf{A})(\mathbf{B})}L^{(\mathbf{C})(\mathbf{D})}\},
\end{aligned}
\tag{3.32}
$$

so that, the line element assumes the form

$$
ds^2=\exp(2\sigma)\eta_{(\mathbf{A})(\mathbf{B})}dz^{(\mathbf{A})}dz^{(\mathbf{B})}. \tag{3.33}
$$

It is conformal to an n-dimensional flat manifold, as well as to an n-dimensional manifold of constant curvature, when, in normal coordinates, they are well-behaved in the origin and in its neighborhood. In this paper, for general relativity, we have $n=4$. In this case there is a time τ and (3.33) can be written in the particular form, as follows

$$
\begin{aligned}
ds^2 = \eta_{(\mathbf{a})(\mathbf{b})} dz^{(\mathbf{a})} dz^{(\mathbf{b})} + \\
+\{\eta_{(\mathbf{0})(\mathbf{0})} + \frac{1}{2}[\frac{1}{2}\epsilon_{(\mathbf{B})} B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} + \\
+\eta^{(\mathbf{M})(\mathbf{N})} A_{(\mathbf{M})(\mathbf{B})(\mathbf{A})} A_{(\mathbf{N})(\mathbf{C})(\mathbf{D})}]. \\
.(z^{(\mathbf{B})}\frac{dz^{(\mathbf{A})}}{d\tau} - z^{(\mathbf{A})}\frac{dz^{(\mathbf{B})}}{d\tau})(z^{(\mathbf{C})}\frac{dz^{(\mathbf{D})}}{d\tau} - z^{(\mathbf{D})}\frac{dz^{(\mathbf{C})}}{d\tau})\} d\tau^2,
\end{aligned}
\tag{3.34}
$$

where$(a), (b) \neq 0$.
Defining

$$
d\rho^2 = \{\eta_{(\mathbf{0})(\mathbf{0})} + \frac{1}{2}[\frac{1}{2}\epsilon_{(\mathbf{B})} B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} + \tag{3.35}
$$

$$
\begin{aligned}
+\eta^{(\mathbf{M})(\mathbf{N})} A_{(\mathbf{M})(\mathbf{B})(\mathbf{A})} A_{(\mathbf{N})(\mathbf{C})(\mathbf{D})}]. \\
.(z^{(\mathbf{B})}\frac{dz^{(\mathbf{A})}}{d\tau} - z^{(\mathbf{A})}\frac{dz^{(\mathbf{B})}}{d\tau})(z^{(\mathbf{C})}\frac{dz^{(\mathbf{D})}}{d\tau} - z^{(\mathbf{D})}\frac{dz^{(\mathbf{C})}}{d\tau})\} d\tau^2,
\end{aligned}
\tag{3.36}
$$

then, (3.34) can be rewritten as

$$
ds^2 = d\rho^2 + \eta_{(\mathbf{a})(\mathbf{b})} dz^{(\mathbf{a})} dz^{(\mathbf{b})}, \tag{3.37}
$$

where in the coordinates ρ and $z^{(\mathbf{a})}$ (3.37) is a flat metric. Note that this is true where the Riemannian normal coordinates are well-behaved in the origin and in its neighborhood.

4 Mapping Among Manifolds

In this section we present a short review of a modification of the Hamiltonian formalism, obtained from a rupture with the symplectic hypothesis [4]. For an extension of the modified Hamiltonian formalism, see [5]. Mapping among manifolds are possible in this modified formalism.

It is well-known that in the Hamiltonian formalism the Hamilton equations and the Poisson brackets will be conserved only by a canonical or symplectic transformation. In the modified-Hamiltonian formalism only Hamilton equations will be conserved, in the sense that they will be transformed into other Hamilton equations by a non-canonical or non-symplectic transformation, and the Poisson brackets will not be invariant. We now build a modified Hamiltonian formalism. Consider a time-dependent Hamiltonian $H(\tau)$ where τ is an affine parameter, in this case, the proper-time of the particle. Let us define 2n variables that will be called ξ^j with index j running from 1 to 2n so that we have $\xi^j \in (\xi^1, \ldots, \xi^n, \xi^{n+1}, \ldots, \xi^{2n}) = (q^1, \ldots, q^n, p^1, \ldots, p^n)$ where q^j and p^j are coordinates and momenta, respectively. We now define the Hamiltonian by

$$H(\tau) = \frac{1}{2} H_{ij} \xi^i \xi^j, \tag{4.1}$$

where H_{ij} is a symmetric matrix. We impose that the Hamiltonian obeys the Hamilton equation

$$\frac{d\xi^i}{d\tau} = J^{ik} \frac{\partial H}{\partial \xi^k}. \tag{4.2}$$

The equation (4.2) introduces the symplectic J, given by

$$\begin{pmatrix} O & I \\ -I & O \end{pmatrix} \tag{4.3}$$

where O and I are the $n\mathbf{x}n$ zero and identity matrices, respectively. We now make a linear transformation from ξ^j to η^j given by

$$\eta^j = T^j{}_k \xi^k, \tag{4.4}$$

where $T^j{}_k$ is a non-symplectic matrix, and the new Hamiltonian is given by

$$Q = \frac{1}{2} C_{ij} \eta^i \eta^j, \tag{4.5}$$

where C_{ij} is a symmetric matrix. The matrices H, C, and T obey the following system

$$\frac{dT^i{}_j}{d\tau} + \frac{dt}{d\tau} T^i{}_k J^{kl} X_{lj} = J^{im} Y_{ml} T^l{}_j, \tag{4.6}$$

where $2X_{lj} = \frac{\partial H_{ij}}{\partial \xi^l} \xi^i + 2H_{lj}$ and $2Y_{ml} = \frac{\partial C_{il}}{\partial \eta^m} \eta^i + 2C_{ml}$, t and τ are the proper-times of the particle in two different manifolds. We note that (4.6) is a first order linear differential equation system in $T^i{}_k$, and it is the response to what we looked for because the non-linearity in the Hamilton equations were transferred to their coefficients. Consider $\frac{dt}{d\tau} X_{lj} = Z_{lj}$ and write (4.6) in the matrix form

$$\frac{dT}{d\tau} + TJZ = JYT, \tag{4.7}$$

where T, Z and Y are $2n\mathbf{x}2n$ matrices as

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \tag{4.8}$$

with similar expressions for Z and Y. Let us write (4.7) as follows

$$\dot{T}_1 = Y_3 T_1 + Y_4 T_3 + T_2 Z_1 - T_1 Z_3, \tag{4.9}$$

$$\dot{T}_2 = Y_3 T_2 + Y_4 T_4 + T_2 Z_2 - T_1 Z_4, \tag{4.10}$$

$$\dot{T}_3 = -Y_1 T_1 - Y_2 T_3 + T_4 Z_1 - T_3 Z_3, \tag{4.11}$$

$$\dot{T}_4 = -Y_1 T_2 - Y_2 T_4 + T_4 Z_2 - T_3 Z_4. \tag{4.12}$$

Now consider

$$\dot{S}_1 = Y_3 S_1 + Y_4 S_3, \tag{4.13}$$

$$\dot{S}_2 = Y_3 S_2 + Y_4 S_4, \tag{4.14}$$

$$\dot{S}_3 = -Y_1 S_1 - Y_2 S_3, \tag{4.15}$$

$$\dot{S}_4 = -Y_1 S_2 - Y_2 S_4, \tag{4.16}$$

and

$$\dot{R}_1 = R_2 Z_1 - R_1 Z_3, \tag{4.17}$$

$$\dot{R}_2 = R_2 Z_2 - R_1 Z_4, \tag{4.18}$$

$$\dot{R}_3 = R_4 Z_1 - R_3 Z_3, \tag{4.19}$$

$$\dot{R}_4 = R_4 Z_2 - R_3 Z_4. \tag{4.20}$$

From the theory of first order differential equation systems [4], it is well-known that the system $(4.13)-(4.20)$ has a solution in the region where Z_{lj} and Y_{ml} are continuous functions. In this case, the solution for (4.6) or (4.7) is given by

$$T_1 = (S_1 a + S_2 b)R_1 + (S_1 d + S_2 c)R_3, \tag{4.21}$$

$$T_2 = (S_1 a + S_2 b)R_2 + (S_1 d + S_2 c)R_4, \tag{4.22}$$

$$T_3 = (S_3 a + S_4 b)R_1 + (S_3 d + S_4 c)R_3, \tag{4.23}$$

$$T_4 = (S_3 a + S_4 b)R_2 + (S_3 d + S_4 c)R_4, \tag{4.24}$$

where a,b,c and d are constant $n\mathbf{x}n$ matrices, and substituting $(4.21)-(4.24)$ in (4.4) we will have completed the mapping among manifolds.

Although this approach is much more general than the one in Section 3, we will need the latter, because mapping, embedding, and other operations are easier in Riemannian normal coordinates.

5 The Mapping

In this section we will build a mapping between two metrics. These two metrics will be (2.8) and another one representing the ordinary matter. When a Riemannian normal coordinate is well-behaved in the origin and in its neighborhood, the metric associated with the ordinary matter can be put in the form (3.33) as follows

$$ds^2 = \exp(2\sigma)\eta_{(\mathbf{A})(\mathbf{B})}dz^{(\mathbf{A})}dz^{(\mathbf{B})}. \tag{5.1}$$

Note that the metric (2.8) can be put in the form (5.1) as follows

$$T = \int e^{-\frac{1}{2}[-2\in mb(\sqrt{\frac{A}{3}})t - \frac{m^2}{3}t^2]}dt, \tag{5.2}$$

$$\Psi = e^{\frac{1}{2}[-2\in mb(\sqrt{\frac{A}{3}})t - \frac{m^2}{3}t^2]}, \tag{5.3}$$

so that (2.8) assume the following form

$$ds^2 = \Psi^2[dT^2 - d\sigma^2]. \tag{5.4}$$

Using the results of section 4 we will build the mapping between (5.1) and (5.4). Note, from (4.4), the dependence of the coordinates and momenta associated with the ordinary matter from the coordinates and momenta associated with (2.8).

6 The Embedding

The embedding of only one metric in a flat space is well known, [8]. For the embedding of (2.8) and a classical metric in a 6-dimensional flat space we proceed as follows.
Let us write the metric associated with the ordinary matter in the form (3.33)

$$ds^2 = \exp(2\sigma)\eta_{(\mathbf{A})(\mathbf{B})}dz^{(\mathbf{A})}dz^{(\mathbf{B})}. \tag{6.1}$$

For the ordinary matter metric (6.1), we define the following transformation of coordinates

$$y^{(\mathbf{A})} = \exp(\sigma)z^{(\mathbf{A})}, \tag{6.2}$$

with $(A) = (1, 2, 3, 4)$,

$$y^5 = \exp(\sigma)(\eta_{(\mathbf{A})(\mathbf{B})}z^{(\mathbf{A})}z^{(\mathbf{B})} - \frac{1}{4}), \tag{6.3}$$

and,

$$y^6 = \exp(\sigma)(\eta_{(\mathbf{A})(\mathbf{B})}z^{(\mathbf{A})}z^{(\mathbf{B})} + \frac{1}{4}). \tag{6.4}$$

It is easy to verify that

$$\eta_{\mathbf{AB}}y^{\mathbf{A}}y^{\mathbf{B}} = 0, \tag{6.5}$$

where,

$$\eta_{\mathbf{AB}} = (\eta_{(\mathbf{A})(\mathbf{B})}, \eta_{(\mathbf{5}),(\mathbf{5})}, \eta_{(\mathbf{6}),(\mathbf{6})}), \tag{6.6}$$

with,

$$\eta_{(\mathbf{5}),(\mathbf{5})} = 1, \tag{6.7}$$

and,

$$\eta_{(\mathbf{6}),(\mathbf{6}) = -\mathbf{1}}. \tag{6.8}$$

By a simple calculation we can verify that the line elements are given by

$$ds^2 = \exp(2\sigma)\eta_{(\mathbf{A})(\mathbf{B})}dz^{(\mathbf{A})}dz^{(\mathbf{B})} = \eta_{\mathbf{AB}}dy^{\mathbf{A}}dy^{\mathbf{B}}. \tag{6.9}$$

The equation (6.5) is a hyper-cone in the (6)-dimensional flat manifold. The metric (6.1) was embedded in the hyper-cone (6.5) of the (6)-dimensional flat manifold. In this paper $n = 4$, so that the hyper-cone represents a region in a 6-dimensional flat manifold, as the light-cone in the Minkowski's spacetime, although, with a different physical meaning.
For the metric (2.8) we have the following transformation of coordinates

$$y'^{1} = \Psi.T, \tag{6.10}$$

$$y'^{\mathbf{i+1}} = \Psi.x^{\mathbf{i}}, \tag{6.11}$$

where $i = (1, 2, 3)$,

$$y'^{5} = \Psi(\eta_{\alpha\beta}x^{\alpha}x^{\beta} - \frac{1}{4}), \tag{6.12}$$

and,

$$y'^{6} = \Psi(\eta_{\alpha\beta}x^{\alpha}x^{\beta} + \frac{1}{4}). \tag{6.13}$$

It is easy to verify that

$$\eta_{\alpha\beta}y'^{\alpha}y'^{\beta} = 0, \tag{6.14}$$

where,

$$\eta_{\alpha\beta} = (\eta_{11}, ..., \eta_{\mathbf{55}}, \eta_{\mathbf{66}}), \tag{6.15}$$

with, $\eta_{\mathbf{11}} = \eta_{\mathbf{55}} = 1$, and $\eta_{\mathbf{22}} = \eta_{\mathbf{33}} = \eta_{\mathbf{44}} = \eta_{\mathbf{66}} = -1$.
By a simple calculation we can verify that the line elements are given by

$$ds^2 = dt^2 - d\sigma^2 e^{[-2\in mb(\sqrt{\frac{A}{3}})t - \frac{m^2}{3}t^2]} = \eta_{\alpha\beta}dy'^{\alpha}dy'^{\beta}. \tag{6.16}$$

Using the results of section 4 we can build the mapping between (6.9) and (6.16). Note, from (4.4), the dependence of the coordinates and the momenta associated with the ordinary matter from coordinates and momenta associated with (2.8). This is more evident from the system (4.21) − (4.24). Note that the metrics are in different regions of the 6-dimensional flat manifold, and the coordinates in different regions of the hyper-cone.

7 Mapping Among Hyper-Vectors

In this section we present the two 6-dimensional hyper-vectors, normal to the manifold (2.8). They can define the directions of the matter-energy flows between the two manifolds embedded in the hyper-cone. Some conventions and results of [9] will be used.
Rewrite (6.10), (6.11), (6.12) and (6.13)

$$y'^{\mathbf{1}} = \Psi.T, \tag{7.1}$$

$$y'^{\mathbf{i+1}} = \Psi.x^{\mathbf{i}}, \tag{7.2}$$

where $i = (1, 2, 3)$, and

$$y'^{\mathbf{5}} = \Psi[T^2 - (x^{\mathbf{1}})^2 - (x^{\mathbf{2}})^2 - (x^{\mathbf{3}})^2 - \frac{1}{4}], \tag{7.3}$$

$$y'^{\mathbf{6}} = \Psi[T^2 - (x^{\mathbf{1}})^2 - (x^{\mathbf{2}})^2 - (x^{\mathbf{3}})^2 + \frac{1}{4}]. \tag{7.4}$$

Let us call ${\eta'_{(a)}}^{\alpha}$ the two 6-dimensional hyper-vectors, normal to the manifold (2.8), where $(a) = (1, 2)$, and $\alpha = (1, 2, 3, 4, 5, 6)$. By a simple but long calculation we obtain [9],

$${\eta'_{(1)}}^{1} = -r[\frac{a}{2} - b't] + [\frac{1}{2r} - \frac{r}{2}(\frac{a}{2} - b't)^2]y'^1, \tag{7.5}$$

$${\eta'_{(1)}}^{i} = [\frac{1}{2r} - \frac{r}{2}(\frac{a}{2} - b't)^2]y'^{i+1}, \tag{7.6}$$

$${\eta'_{(1)}}^{k} = [\frac{1}{2r} - \frac{r}{2}(\frac{a}{2} - b't)^2]y'^k - 2r[(\frac{a}{2} - b't)T + \dot{T}], \tag{7.7}$$

and

$${\eta'_{(2)}}^{1} = r[\frac{a}{2} - b't] + [\frac{1}{2r} + \frac{r}{2}(\frac{a}{2} - b't)^2]y'^1, \tag{7.8}$$

$${\eta'_{(2)}}^{i} = [\frac{1}{2r} + \frac{r}{2}(\frac{a}{2} - b't)^2]y'^{i+1}, \tag{7.9}$$

$${\eta'_{(2)}}^{k} = [\frac{1}{2r} + \frac{r}{2}(\frac{a}{2} - b't)^2]y'^k - 2r[(\frac{a}{2} - b't)T + \dot{T}], \tag{7.10}$$

where $i = (1,2,3)$, $k = (5,6)$, $a = -2 \in mb(\sqrt{\frac{A}{3}})$, $b' = \frac{m^2}{3}$, $\dot{T} = \frac{dT}{dt}$, and $r = r(t)$ is given by

$$(\frac{a}{2} - b't)^2 = \exp{(b'r^2)}[c + 2b' \int \frac{1}{r} \exp{(-b'r^2)} dr], \tag{7.11}$$

and c is an integration constant.
From simple calculation we can verify the following conditions,

$$\eta_{\mathbf{AB}} {\eta'_{(1)}}^{\mathbf{A}} {\eta'_{(1)}}^{\mathbf{B}} = 1, \tag{7.12}$$

$$\eta_{\mathbf{AB}} {\eta'_{(2)}}^{\mathbf{A}} {\eta'_{(2)}}^{\mathbf{B}} = -1. \tag{7.13}$$

Note that the two 6-dimensional hyper-vectors (7.12) and (7.13), normal to the manifold (2.8), live in the 6-dimensional flat manifold, while 6-dimensional hyper-vectors, as (6.5) and (6.14), live in the hyper-cone. It is possible that the hyper-vectors associated with the ordinary matter manifold, represented by (6.9), could be integrable. In this case, from Section 4, we can build a Hamiltonian function to each hyper-vector and the mapping between them. Note, from (4.4), the coordinate and momentum dependence between the ordinary matter metric and (2.8), where ${\eta_{(a)}}^{\alpha}$ and ${\eta'_{(a)}}^{\alpha}$ are coordinates associated with the ordinary matter metric and (2.8), respectively. The hyper-vectors can define the directions of the matter-energy flows between the two embedded manifolds. From (4.4) we conclude that the mapping connects these flows.

8 Concluding Remarks

Physicists and astronomers are convinced of the dark matter and dark energy existence. There is a great interest in it and many researchers have concentrated their efforts in verifying and solving such hypothesis with a view point different from the one presented in this paper, [9], [10], [11], [12], [13], [14]. This motivates the emergence of new methods, new formalisms. In this paper we have used an extended Cartan's approach, where pseudo-Riemannian metrics are conformal to flat manifolds when Riemannian normal coordinates are well-behaved in the origin and in its neighborhood. We also presented a modified Hamiltonian formalism, where the symplectic hypothesis was abandoned. This formalism allows us to make the mapping among geometric objects as metrics and hyper-vectors, for instance.

It is important to pay attention to the fact that a Riemannian normal transformation and its inverse are well-behaved in the region where geodesics are not mixed. Points where geodesics close or mix are known as conjugate points of Jacobi's fields. Jacobi's fields can be used for this purpose, [3]. After we put two metrics in the conformal flat form, we accomplish the map between them, obtaining (4.4), where coordinates and momenta will be connected, exerting a mutual gravitational influence. We made the same procedure for the hyper-vectors. An interesting application could be a mapping between (2.8) and the Schwarzschild's metric. In general (4.4) is invertible, and it enables an analysis of the gravitational influence between the metrics. Another interesting investigation would be a mapping between (2.8) and a galaxy. Also an interesting investigation would be a mapping between two hyper-vectors. It is possible that the hyper-vectors associated with the ordinary matter manifold could be integrable, obtaining the two 6-dimensional hyper-vectors, normal to the ordinary matter manifold. In this case, from section 4, we can build a Hamiltonian function for each hyper-vector and the mapping between them. Note, from (4.4), the coordinate and momentum dependence between the ordinary matter metric and (2.8), where ${\eta_{(a)}}^{\mathbf{A}}$ and ${\eta'_{(a)}}^{\mathbf{A}}$ are coordinates associated with the ordinary matter metric and (2.8), respectively. The hyper-vectors can define the directions of the matter-energy flows between the two embedded manifolds. From (4.4) we conclude that the mapping connects the flows. We do not know if those mapping represent a physical reality, as the matter-energy flows between the two embedded manifolds. The formalism presented in Section 4 allows us to make

mapping among different metrics, different geometric objects, etc. However, our choice of a 6-dimensional flat manifold was motivated by the possibility that normal hyper-vectors could define the direction of matter-energy flows between two embedded manifolds, if these flows there exist.

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There is a small mistake in the equation (47.4) which spreads along some Sections. It is important to correct it. In (47.4), replace $\epsilon_{(\sigma)}$ by one. This implies a small change in other equations, as (47.5) and (47.6). In other words, $\epsilon_{(\sigma)}$ will have to appear in (47.5) and (47.6), for instance.

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